Expander Graphs
Exercise Sheet 5

Question 1. Show that Theorem 6.5 (or it's proof) implies that there exists a $(d^4, d, \frac{1}{4})$-graph for some large constant d.

Question 2. Let $\alpha, \beta < 1$, let G be an (n, m, α)-graph and let H be and (m, d, β)-graph. Show that $G \textcircled{z} H$ is an $(nm, d^2, \phi(\alpha, \beta))$-graph where $\phi < 1$.

(Hint: Consider an arbitrary $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h} \perp \boldsymbol{u}$ as in the lecture and aplit into cases as to whether $\boldsymbol{g}$ or $\boldsymbol{h}$ are large.)

Question 3. Let A be the adjacency matrix of an (n, d, λ)-graph and let J be a matrix with every entry equal to $\frac{1}{n}$. Show that $A = (1-\lambda)J + \lambda C$ where the operator norm

$$\sup_{||v||_2=1} ||\boldsymbol{v}C||_2 := ||C|| \leq 1.$$

Let G be an (n, m, α)-graph and let H be and (m, d, β)-graph. Show that $G \textcircled{z} H$ is an $(nm, d^2, \phi(\alpha, \beta))$-graph where ϕ satisfies

$$\phi(\alpha, \beta) \leq 1 - (1-\alpha)(1-\beta)^2$$

Question 4. Show that every connected, non-bipartite (n, d)-graph is such that it's generalised second eigenvalue satisfies

$$\lambda(G) \leq 1 - \frac{1}{dn^2}.$$

As an alternative analysis of the randomised algorithm for USTCON, show that for any two vertices s and t in the same non-bipartite component of a graph $G = (V, E)$ the expected number of steps in a random walk starting at s before we hit t is at most $2|V| \cdot |E|$.

Deduce that a random walk of length $\Omega(n^3)$ starting at s will hit t with probability at least $\frac{1}{2}$.

Question 5. Let H be a finite group and let S be a symmetric generating set of H. The *Cayley graph* $G(H, S)$ is the graph whose vertex set is H and where we have an edge (h, hs) for each $h \in G$ and $s \in S$.

Let $\chi : H \to \mathcal{C}$ be a character of H. Show that the vector $\boldsymbol{v} \in \mathbb{R}^V$ given by $v_h = \chi(h)$ is an eigenvector of G with eigenvalue given by

$$\frac{1}{|S|} \sum_{s \in S} \chi(s).$$

Determine the spectrum of the hypercube Q^d.